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Massive spheres with an isothermal core

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Abstract. Massive spherical configurations with an isothermal core and an inverse-square variation of density in the envelope have been studied. It is found that such structures are pulsationally stable. Keeping in view the relativistic restriction on the speed of a signal ($v_s \leq c$), it is found that $K_{max} = 0{\cdot}80$ for such structures. The maximum surface and central red shifts are found to be 0·502 and 28·7 respectively. An application of such structures to the case of a neutron star gives the maximum mass and size of the star to be $2{\cdot}31 M_{\odot}$ and 12·2 km.

1. Introduction

Exact solutions for the isothermal equation of state ($P = K\rho$) were first obtained by Misner and Zapolsky (1964) for infinite central densities. Brecher and Caporosso (1976) applied these solutions to isothermal neutron star cores. Recently Durgapel *et al* (1979) have determined general solutions for the isothermal equation of state and applied them to estimate the mass and size of neutron stars with different isothermal cores. They have also shown that the solutions as given by Misner and Zapolsky (1964) pertain to a special case of their solutions, namely when the central density, ρ_c, becomes infinite.

Solutions for spherical systems with isothermal cores corresponding to $K = 1/3$ and an envelope having $(dP/d\rho) = 1$ were determined by Bondi (1964). Das and Narlikar (1976) extended Bondi's work to isothermal cores with different values of $K \leq 0{\cdot}5$ and $(dP/d\rho) \leq 1$. Bondi (1964) and Das and Narlikar (1976) also considered envelopes with constant density, but such solutions are unrealistic and unlikely to occur in nature because constant density implies $(dP/d\rho) = -\infty$ and hence an infinite and imaginary value for speed of sound ($v_s = (dP/d\rho)^{1/2}$). Further, no calculations have been done for the stability under radial pulsation for structures with envelopes of constant $(dP/d\rho)$. Yabushita (1977) has shown that the solutions corresponding to maximum red shifts (Das and Narlikar 1976) are not stable under radial pulsations. Massive spherical systems have also been discussed by Durgapal and Gehlot (1969) and Gehlot and Durgapal (1971), who considered a core of constant density surrounded by an envelope of varying density ($\rho \propto r^{-2}$). In their solutions it is not certain whether such systems are stable under radial pulsations, and also whether a constant-density core pertains to infinite and imaginary velocity of sound in the core.

In this paper we have constructed a more realistic model which can be applied to physical problems. The isothermal core is surrounded by an envelope having a density

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$\rho \propto r^{-2}$. We have obtained exact solutions throughout the sphere by matching the two solutions

$$P = K\rho, \qquad 0 \leq r \leq b \text{ (core)}$$

$$8\pi\rho = C/r^2, \qquad b \leq r \leq a \text{ (envelope)}$$

at the boundary $r = b$. This study has been made for various values of K.

In addition to finding exact solutions for the core and envelope system described above, we have also studied the variation of P, ρ, $(\mathrm{d}P/\mathrm{d}\rho)$ with r. Moreover, the stability of these systems under radial pulsations, following the method given by Chandrasekhar (1964) and Harrison *et al* (1965) has also been studied.

Based on this model, the size and mass of neutron stars has been calculated by considering the density at $r = a$ equal to $2 \times 10^{14}\,\mathrm{g\,cm^{-3}}$, and their radial pulsation frequencies have also been calculated following Harrison *et al* (1965). Lastly, the red shifts have been calculated at the surface ($r = a$), the boundary ($r = b$) and the centre ($r = 0$). Under physically realisable conditions the maximum central red shift, Z_c, is found to be 29·8, which is by far the highest for a physically realistic system.

The general assumptions made for solving Einstein's field equations are the following:

(a) The core ($0 \leq r \leq b$), which corresponds to an equation of state $P = K\rho$, is surrounded by an envelope with a density distribution $\rho \propto 1/r^2$.

(b) The system is spherically symmetric and static, and the space is empty outside a finite region of radius a.

(c) At $r = a$, the interior and exterior Schwarzschild solutions should match, that is, $\mathrm{e}^{\nu_a} = \mathrm{e}^{-\lambda_a} = 1 - (2m/a)$ and $P(r = a) = 0$.

(d) The solutions (or the values of ν, λ, P and ρ) in the core and the envelope have the same values at the internal boundary $r = b$.

(e) The pressure and density are everywhere positive and finite. The density at all points decreases with increasing r. The ratio P/ρ at no point increases with increasing r.

(f) The velocity of sound never exceeds the velocity of light, i.e. $(\mathrm{d}P/\mathrm{d}\rho)^{1/2} \leq 1$.

(g) The structures are stable under radial pulsations.

Taking the velocity of light c equal to 1 and the gravitational constant G equal to 1, the relations between the energy density ρ, the pressure P, and the energy-momentum tensor of a perfect fluid are given by

$$\rho = T_0^0, \qquad P = -T_1^1 = -T_2^2 = -T_3^3. \tag{1}$$

2. Field equations and their solutions

2.1. *Field equations*

The line element is given by

$$\mathrm{d}s^2 = \mathrm{e}^{\nu}\,\mathrm{d}t^2 - \mathrm{e}^{\lambda}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\phi^2. \tag{2}$$

Here, λ and ν are functions of r alone. The resulting field equations are

$$-8\pi T_1^1 = 8\pi P = \mathrm{e}^{-\lambda}(\nu/r + 1/r^2) - 1/r^2, \tag{3}$$

$$-8\pi T_2^2 = -8\pi T_3^3 = 8\pi P = \mathrm{e}^{-\lambda}[\nu''/2 + \nu'^2/4 - \lambda'\nu'/4 + (\lambda' - \nu')/2r], \tag{4}$$

$$8\pi T_0^0 = 8\pi\rho = 1/r^2 + \mathrm{e}^{-\lambda}(\lambda'/r - 1/r^2), \tag{5}$$

where primes denote differentiation with respect to r. From equations (3), (4) and (5) we obtain (Oppenheimer and Volkoff 1939)

$$P' = -(P+\rho)\nu'/2, \tag{6}$$

$$m(r) = \int_0^r 4\pi\rho r^2 \, \mathrm{d}r, \tag{7}$$

$$P' = -\frac{(P+\rho)[4\pi P r^3 + m(r)]}{r[r-2m(r)]}. \tag{8}$$

2.2. *The solutions for the core*

$0 \leqslant r \leqslant b$: The equation of state is given by

$$P = K\rho. \tag{9}$$

Using equations (6) and (9), we obtain

$$\mathrm{e}^{-\nu} = A\rho^{2K/(K+1)}. \tag{10}$$

The Oppenheimer–Volkoff equation (8) for $P = K\rho$ can be written as

$$1-(8\pi/r)\int_0^r \rho r^2 \, \mathrm{d}r = (8\pi\rho r^2+1)\left[1-\frac{2K}{K+1}\left(\frac{\rho' r}{\rho}\right)\right]^{-1}. \tag{11}$$

Equation (11) can be computed to evaluate the density and e^{λ} within the core following the procedure as given by Durgapal *et al* (1979) for an isothermal massive sphere. We assume the central density ρ_c to be

$$\rho_c = 3K/2\pi(K+1)(3K+1)r_0^2, \tag{12}$$

where r_0 is an interval depending upon the choice of the central density. Equation (11) is now matched at successive points for the value of r in the range $0 \leqslant r \leqslant 6r_0$. The value of $\rho/\rho_c (= P/P_c)$ and $2m(r)/r$ for different values of $r = Nr_0$ are shown in table 1 for $K = 0{\cdot}2$, $\frac{1}{3}$, $0{\cdot}4$, $0{\cdot}6$, $0{\cdot}8$ and $1{\cdot}0$ respectively.

2.3. *Solutions in the envelope*

$b \leqslant r \leqslant a$. The density is given by

$$8\pi\rho = C/r^2 \tag{13}$$

The solutions for $C \leqslant \frac{1}{2}$ are given by (Gehlot and Durgapal 1971)

$$\mathrm{e}^{-\lambda} = 1 - C = 1-(2m/a), \tag{14}$$

$$\mathrm{e}^{\nu/2} = [(1+n)^2(r/a)^{1-n} - (1-n)^2(r/a)^{1+n}]/4n(2-n^2)^{1/2}, \tag{15}$$

$$8\pi P = \frac{C(1-n^2)}{r^2}\frac{[(r/a)^{-n}-(r/a)^n]}{[(1+n)^2(r/a)^{-n}-(1-n)^2(r/a)^n]}, \tag{16}$$

where $n^2 = (1-2C)/(1-C)$.

Table 1.

N ($= r/r_0$)	$K = 0{\cdot}2$ ρ/ρ_c	$2m(r)/r$	$K = \frac{1}{3}$ ρ/ρ_c	$2m(r)/r$	$K = 0{\cdot}4$ ρ/ρ_c	$2m(r)/r$	$K = 0{\cdot}6$ ρ/ρ_c	$2m(r)/r$
$\frac{1}{8}$	0·9840	0·008059	0·9843	0·0077	0·9843	0·008000	0·9843	0·01629
$\frac{1}{4}$	0·9835	0·02506	0·9386	0·0301	0·9317	0·0313	0·9180	0·03622
$\frac{3}{8}$	0·8680	0·05381	0·8682	0·0646	0·8678	0·06710	0·8668	0·07181
$\frac{1}{2}$	0·7810	0·08980	0·7809	0·1078	0·7802	0·1119	0·7791	0·1173
$\frac{5}{8}$	0·6861	0·1298	0·6849	0·1557	0·6804	0·1751	0·6838	0·1681
$\frac{3}{4}$	0·5899	0·1708	0·5918	0·2050	0·5823	0·2234	0·5888	0·2204
$\frac{7}{8}$	0·5008	0·2105	0·5020	0·2527	0·4913	0·2701	0·4995	0·2711
1	0·4232	0·2473	0·4200	0·2968	0·4016	0·3123	0·4188	0·3179
$\frac{5}{4}$	0·2938	0·3090	0·2893	0·3693	0·2785	0·3814	0·2906	0·3958
$\frac{3}{2}$	0·2046	0·3539	0·2000	0·4211	0·1942	0·4317	0·2026	0·4524
$\frac{7}{4}$	0·1446	0·3844	0·1406	0·4557	0·1380	0·4661	0·1438	0·4910
2	0·1051	0·4040	0·1015	0·4774	0·1003	0·4894	0·1046	0·5161
$\frac{5}{2}$	0·06066	0·4237	0·05850	0·4982	0·05967	0·5147	0·0609	0·5420
3	0·03823	0·4286	0·03670	0·5024	0·03854	0·5034	0·03882	0·5497
$\frac{7}{2}$	0·0259	0·4265	0·02500	0·4990	0·02546	0·5064	0·02662	0·5488

N ($= r/r_0$)	$K = 0{\cdot}8$ ρ/ρ_c	$2m(r)/r$	$K = 1{\cdot}0$ ρ/ρ_c	$2m(r)/r$	$K = 2{\cdot}0$ ρ/ρ_c	$2m(r)/r$
$\frac{1}{8}$	0·9843	0·008088	0·9844	0·0301	0·9844	0·005896
$\frac{1}{4}$	0·9388	0·03147	0·9390	0·0646	0·9396	0·02293
$\frac{3}{8}$	0·8687	0·06757	0·8693	0·1079	0·8714	0·04930
$\frac{1}{2}$	0·7821	0·1128	0·7834	0·1562	0·7881	0·08247
$\frac{5}{8}$	0·6878	0·1631	0·6900	0·2060	0·6984	0·1149
$\frac{3}{4}$	0·5937	0·2149	0·5970	0·2546	0·6096	0·1585
$\frac{7}{8}$	0·5051	0·2651	0·5094	0·2999	0·5262	0·1969
1	0·4255	0·3118	0·4308	0·3769	0·4511	0·2334
$\frac{5}{4}$	0·2977	0·3903	0·3041	0·4347	0·3289	0·2977
$\frac{3}{2}$	0·2092	0·4483	0·2156	0·4760	0·2413	0·3492
$\frac{7}{4}$	0·1498	0·4887	0·1551	0·5044	0·1798	0·3889
2	0·1095	0·5156	0·1152	0·5373	0·1367	0·4188
$\frac{5}{2}$	0·06433	0·5450	0·06820	0·5508	0·08444	0·4585
3	0·04124	0·5555	0·04400	0·5546	0·05601	0·4806
$\frac{7}{2}$	0·02840	0·5568	0·03040	0·5534	0·03940	0·4926
4					0·02906	0·4987
5					0·01772	0·5029
6					0·01182	0·5060

The solutions for $C \geqslant \frac{1}{2}$ are

$$e^{\nu/2} = e^{-x}[2k\cos(kx) + (1-k^2)\sin(kx)]/2k(2+k^2)^{1/2}, \tag{17}$$

$$8\pi P = C(1+k^2)/r^2[(1-k^2) + 2k\cot(kx)], \tag{18}$$

where $k^2 = (2C-1)/(1-C)$, $x = \ln(a/r)$.

2.4. Continuity at the boundary $r = b$

At the boundary $r = b$, we have

$$P = K\rho_b$$

or

$$K = (1-n^2)[1-(b/a)^{2n}]/[(1+n)^2-(1-n)^2(b/a)^{2n}] \quad \text{for } C \geq \tfrac{1}{2}$$
$$= (1+k^2)[(1-k^2)+2k\cot(kx_b)]^{-1} \quad \text{for } C \geq \tfrac{1}{2} \tag{19}$$

where $x_b = \ln(a/b)$.

Continuity of $e^{-\lambda}$ and ρ at $r = b$ gives

$$8\pi\rho_b b^2 = 2m(b)/b = 2m/a$$

or

$$N^2(\rho_b/\rho_c) = \frac{K}{6(K+1)(3K+1)}\frac{m(b)}{b}. \tag{20}$$

Continuity of e^{ν} and ρ gives the value of the constant A in the expression (10). Using equation (20) and table 1, we can find the value of $m(b)/b = C/2$ which in turn gives

$$n^2 = -k^2 = (1-2C)/(1-C). \tag{21}$$

Once the value of n or k is known we can find the value of (b/a). For $C \leq \frac{1}{2}$

$$b/a = \left\{\left[1-K\left(\frac{1+n}{1-n}\right)\right]\bigg/\left[1-K\left(\frac{1-n}{1+n}\right)\right]\right\}^{1/2n}. \tag{22a}$$

For $C \geq \frac{1}{2}$

$$x_b = \ln(a/b) = (1/k)\tan^{-1}\left[\frac{2kK}{(1-K)+(1+K)k^2}\right]. \tag{22b}$$

Calculated values of (ρ_c/ρ_a), (b/a), (b/r_0) and C are given in table 2.

Table 2.

K	ρ_c/ρ_a	b/a	b/r_0	C
0·2	142·6	0·4283	3·00	0·4285
$\frac{1}{3}$	192·6	0·3761	3·00	0·5024
0·4	503·4	0·2793	3·50	0·5063
0·6	395·6	0·2552	3·00	0·5498
0·8	801·4	0·1739	3·00	0·5569
1·0	2933	0·1059	3·50	0·5545
2·0	∞	0	5·00	0·5030

3. Model of neutron stars

3.1. Mass and radius

Assuming the density ρ_a at $r = a$ to be 2×10^{14} g cm^{-3}, the masses of the configurations are calculated. The values of r_0, a, b, m_{core}, m_{envelope}, m and $(dP/d\rho)^{1/2}$ at $r = a$ and b are shown in table 3. The value of $(dP/d\rho)$ exceeds unity for values of $K > 0·8$. The only physically realistic models are those which correspond to $K < 0·8$.

Table 3.

K	r_0 (in km)	a (in km)	b (in km)	$m_{core}/M_\odot$	$m_{env}/M_\odot$	$m_{tot}/M_\odot$	$(dP/d\rho)_b$	$(dP/d\rho)_a$
0·2	1·530	10·72	4·590	0·6674	0·8909	1·558	0·5196	0·4329
$\frac{1}{3}$	1·506	11·61	4·518	0·7698	1·235	2·005	0·6699	0·5024
0·4	0·9298	11·65	3·254	0·5589	1·442	2·001	0·7124	0·5063
0·6	1·032	12·14	3·096	0·5773	1·686	2·263	0·8840	0·5525
0·8	0·7085	12·22	2·125	0·4004	1·906	2·306	1·009	0·5605
1·0	0·3689	12·19	1·291	0·2429	2·050	2·293	1·114	0·5578
2·0	0	11·61	0	0	1·981	1·981	1·483	2·289

3.2. *Pulsational stability and frequency*

It has been shown by Chandrasekhar (1964) and Harrison *et al* (1965), using a variational principle, that an equilibrium configuration for a sphere of cold catalysed matter is stable against radial deformations if and only if the potential energy is positive definite. The potential energy Ω is given by the expression

$$\Omega = \int_0^a e^{\lambda/2}\, e^{3\nu/2}[4r^3P' + 9\gamma P r^3 - r^4(P'^2/P+\rho)] + 8\pi \int_0^a e^{3\lambda/2}\, e^{3\nu/2} P(P+\rho) r^4\, dr. \tag{23}$$

Since the solutions in the core and the envelope are known, Ω can be evaluated straightaway. The results are given in table 3 and one can easily see that for all values of K the potential energy is positive definite, implying that such structures are stable against radial deformations.

Harrison *et al* (1965) have also shown that the radial acoustical modes of affective vibrations of a sphere of cold catalysed matter have a normal angular frequency which is given by

$$\omega^2 = \Omega/\Delta \tag{24}$$

where

$$\Delta = \int_0^a e^{3\lambda/2}\, e^{\nu/2}(P+\rho) r^4\, dr. \tag{25}$$

Calculating Δ (for different K) using equation (25), one can thus obtain the frequencies, $f = \omega/2\pi$. The results are given in table 3.

4. Central red shift

An attempt has been made to evaluate the maximum red shift for the structures which are pulsationally stable and in which $dP/d\rho \leq 1$. In § 3.2, we have already seen that the structures discussed in this paper are stable under radial pulsations. The expression for $(dP/d\rho)$ can easily be determined as:

for $C \leq \frac{1}{2}$,

$$v_s = (dP/d\rho)^{1/2} = (1-n^2)^{1/2}[(1-n)(r/a)^{2n} - (1+n)]/[(1+n)^2 - (1-n)^2(r/a)^{2n}]$$

and for $C \geqslant \frac{1}{2}$,

$$v_s = (1+k^2)^{1/2}[1+k\cot(kx)]/[(1-k^2)+2k\cot(kx)].$$

Values of $dP/d\rho$ at $r=a$ and $r=b$ are shown in table 3.

The red shifts Z_a, Z_b, and Z_c at $r=a$, b and c respectively can be determined as follows.

$$1+Z_s = (1-C)^{-1/2},$$

$$1+Z_b = 4n(2-n^2)^{1/2}[(1+n)^2(b/a)^{1-n}-(1-n)^2(b/a)^{1+n}]^{-1} \qquad \text{for } C \leqslant \tfrac{1}{2},$$

$$= (a/b)K(2+k^2)^{1/2}[2k\cos kx_b + (1-k^2)\sin kx_b]^{-1} \qquad \text{for } C \geqslant \tfrac{1}{2}.$$

Using equation (15), we see that

$$\frac{1+Z_c}{1+Z_b} = (\rho_c/\rho_b)^{K/(K+1)}.$$

The values of Z_a, Z_b and Z_c are also given in table 4. The maximum value of red shift which a physically realisable structure can have is found to be 28·72.

Table 4.

K	$\Omega \times 10^{-5}$	$\Delta \times 10^{-15}$	f (in Hz)	Z_a	Z_b	Z_c
0·2	0·05614	11·14	3389	0·323	0·887	2·251
$\frac{1}{3}$	0·02482	25·41	1492	0·418	1·549	4·824
0·4	0·02332	20·07	1628	0·423	2·193	8·112
0·6	0·02842	26·61	1560	0·490	3·505	14·23
0·8	0·02910	28·08	1537	0·502	6·204	28·72
1·0	0·02903	27·63	1548	0·498	12·46	70·44
2·0	0·01810	14·65	1678	0·418	$6·817 \times 10^{10}$	$1·003 \times 10^{12}$

5. Results and discussion

(i) As K is increased it is seen that core size, as compared to the size of the configuration, decreases—when $K = 2·0$, the core size is almost zero (table 2).

(ii) The surface red shift, Z_a, also increases with K, but has a maximum value 0·502 when $K = 0·80$. Beyond this K value it again decreases (table 4).

(iii) On the other hand, the central red shift Z_c steadily increases with K, tending to ∞ as K is increased beyond 1·0. For $K = 0·80$, the central red shift turns out to be 28·72 (table 4).

(iv) These structures with an isothermal core and an inverse-square variation of density in the envelope, are seen to be stable under radial pulsations—Ω turns out to be positive for all the cases. The radial pulsational frequencies are $\sim 10^3$ Hz (table 4).

(v) The values of $(dP/d\rho)$ at $r=b$ and $r=a$ increase with K. For all cases except $K = 2·0$, $(dP/d\rho)_b \geqslant (dP/d\rho)_a$, a condition which must be satisfied for a realistic case. Moreover, since $(dP/d\rho) \leqslant 1$ for the speed of sound not to exceed the speed of light, 0·80 turns out to be the maximum value of K because at this K, $(dP/d\rho)_b$ just becomes unity (table 3).

(vi) Taking $\rho_a = 2\times10^{14}\,\text{g cm}^{-3}$, the masses of the core and envelope have been calculated (table 3). The mass of the core decreases with K, while the mass of the envelope increases up to $K = 1\cdot0$, beyond which it decreases. The maximum total mass is again obtained for $K = 0\cdot80$. Based on this model for a neutron star, the maximum mass is found to be $2\cdot306M_\odot$. The size of the structure, for a neutron star model, is ~ 10 km—the maximum value 12·22 km appearing for $K = 0\cdot80$. The core radius, however, decreases with K.

It is thus seen that for such structures $K = 0\cdot80$ should be considered the maximum K value for a physically realistic case. (Incidently, Durgapal *et al* (1979) have also found that $K = 0\cdot80$ gives the maximum mass for an isothermal sphere.)

Values of $K > 0\cdot80$ pertain to supraluminal cases and hence should not be used for physically real situations. The maximum values for surface and central red shifts (for $K = 0\cdot80$) turn out to be 0·502 and 28·72 respectively—the latter being the maximum value as far as is known to the authors. As applied to the case of a neutron star, the maximum mass is $2\cdot306M_\odot$ and the maximum size is 12·22 km.

The maximisation as done in this study has the additional features that (i) the central density is always finite and (ii) the supraluminal cases have been avoided altogether.

Apart from all this, it is also seen that for supraluminal cases there is an erratic behaviour in the solutions. This is another reason why these cases should be avoided in constructing models pertaining to physical situations.

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